LOWER-BOUND APPROACH TO THE LIMIT ANALYSIS OF 3D VAULTED BLOCK MASONRY STRUCTURES

C. Casapulla[1] and D. D'Ayala[2]

1. ABSTRACT

A review of existing lower bound approaches for dry block masonry structures reveals a lack of reliable analytical methods applicable to most general conditions. Usually the analysis is restricted to cases in which sliding is prevented by high friction among block interfaces. This leads, for arches, to the well-known hinging mechanisms first discussed in terms of plastic analysis by Heyman (1966).

However, especially for historic buildings the quality of the contact surfaces or of the binding materials might be deteriorated so as to substantially reduce the original friction coefficient. In addition, some particular shapes of curvilinear structures, e.g. flat arches, would never collapse unless sliding occurred. Hence the necessity of studying this group of problems under the more realistic assumptions of presence of sliding and absence of dilatancy. Herewith first is presented a proof of uniqueness of the solution for the limit state analysis of 3D masonry arches, in the condition of axial symmetry of geometry and loading. This proof is crucial to the robustness of the results and allows a straightforward treatment of this class of problems as one of standard limit-state analysis. The analysis can easily be extended to barrel vaults, common in historic buildings and forming the structure of masonry bridges.

On these assumptions a simple but very adaptable computer procedure, using a lower bound approach, has been developed for the calculation of the minimum thickness required to ensure stability for such structures. Parametric analysis shows how the geometry of the barrel vault varies depending on the eccentricity and inclination of the applied loads.

2. INTRODUCTION

The original application of plastic analysis to masonry structures proposed by Heyman [1], [2] is based on the assumptions of infinite compressive strength, no tensile strength and infinite friction resistance. In the last two decades numerous studies have applied this model to different types of dry masonry structures and discussed the effects of the relaxation of some or all the original assumptions. Specifically, if the possibility of sliding is contemplated, then the material behaviour is non-standard in terms of plasticity theory.

Keywords: 3D vaulted masonry, Coulomb friction, Limit analysis, Unique solutions

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The satisfaction of the normality rule will imply the presence of dilatancy, which does not occur in reality, and non-compliance with the normality rule means that the fundamental theorems of plasticity will not in general provide a unique solution. A general review of the applicability of limit state analysis to finite friction systems and a first analysis of cases for which a unique solution can be found are contained in [3]. Livesley [4], [5], by adopting a lower-bound approach, was the first to develop a formal linear programming procedure to define the maximum load factor of two and three-dimensional structures. Masonry domes have been studied in more details by D'Ayala [6], [7]. Livesley proposed a post-optimality analysis to test the uniqueness of the computed load factor and concluded that, depending on the geometry and friction coefficient, only for some problems limit state analysis provides the correct lower bound collapse factor. The true mechanism is obtained by a manipulation of the associated kinematical solution. Gilbert & Melbourne [8], [9] have instead adopted a kinematic approach, including dilatancy, to the analysis of 2D multi-ring brickwork arches, under self-weight and live loads.

Mathematical programming has been identified in the past as the best tool to solve the problem and Lo Bianco & Mazzarella [10] have proposed a procedure that satisfies uniqueness and does not imply dilatancy. More recently Baggio & Trovalusci [11], [12] have proposed a non-standard limit analysis approach based on determining the minimum of a class of load factors satisfying the kinematic and static condition simultaneously. The associated programme results however rather onerous in terms of time and memory requirements, as it depends on non-linear and non-convex optimisation procedures.

A conservative approach is represented by the identification of reduced failure domains which lead to solutions statically admissible which satisfy the normality rule [3], [13], [14], [15]. On the other hand it is appropriate to systematically identify classes of problems for which a unique solution can be found. This is the scope of the work presented in this paper.

The uniqueness of the solution, due to the symmetry, has been demonstrated by Sinopoli et al. [16] for 2D masonry arches through a parametric analysis. Here this is generalised to 3D systems by studying the behaviour of the block interface and by a critical analysis of the results obtained by application of a procedure based on optimum search. A previous study of 2D masonry arches was carried out in [13] and a further study of masonry domes is being carried out in [17], both using a similar procedure.

3. 3D DRY BLOCK MASONRY ARCHES

3.1 Uniqueness of the solution

3D arches can be modelled as three-dimensional discrete systems of rigid blocks in dry contact under the assumptions of infinite compressive strength for the blocks, no tension transmitted across the joints, shear strength at the block interfaces determined by cohesionless Coulomb friction.

The governing relations of such systems formally correspond to those of a non-standard rigid-plastic discretised structure, where the block interfaces are treated as the elements of the problem and the blocks are simply defining the geometry of the problem. Therefore the analysis is fully related to the behaviour of the interfaces, which can then be regarded as systems of stresses associated to frictional (plastic) constraints. Safety under the stated material assumptions is assured if a thrust line can be found which lies wholly within the masonry arch and satisfies frictional constraints.

Consider the symmetric 3D arch in Figure 1, subjected to its own weight and to a couple of point loads P, acting generally at some point of it, eccentrically with respect to the mean plane

XZ and symmetrically with respect to the vertical plane YZ. The contact forces between two adjacent voussoirs may have components both normal and tangential to the interface, the latter being resisted by friction. Also torsion moment will arise on the interface in order to equilibrate the action produced by the eccentric application of the external loads.

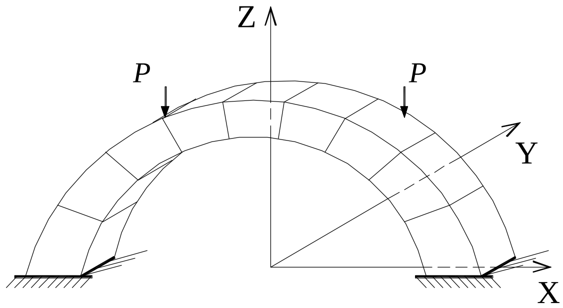

Fig. 1. The 3D voussoir arch

In order to prove the uniqueness of the solution, if reference is made to a condition of vertical loads applied to a generic voussoir, internal and external forces are contained in a vertical plane, identified by the XZ axes in Figure 2.

Let now *H*, be the horizontal reactive thrust on a contact joint at some angle a from the vertical axis Z, *W* the self-weight of the considered upper and symmetric portion of the arch and *P* the point load applied to this portion. By drawing at this interface the projection of the cohesionless Coulomb's cone, it is self-evident that a range of admissible values of *H* can be identified depending on the applied load and on the friction coefficient. Then, being j the friction angle, this range can easily be expressed by:

$$(W+P)\cot(a+j) \leq H \leq (W+P)\cot(a-j) \tag{1}$$

where the lower and upper bound define points A and B on the Coulomb's cone projection respectively (Figure 2), implying the incipient inward and outward sliding of the upper portion of the arch on the lower one when either of the equal signs hold.

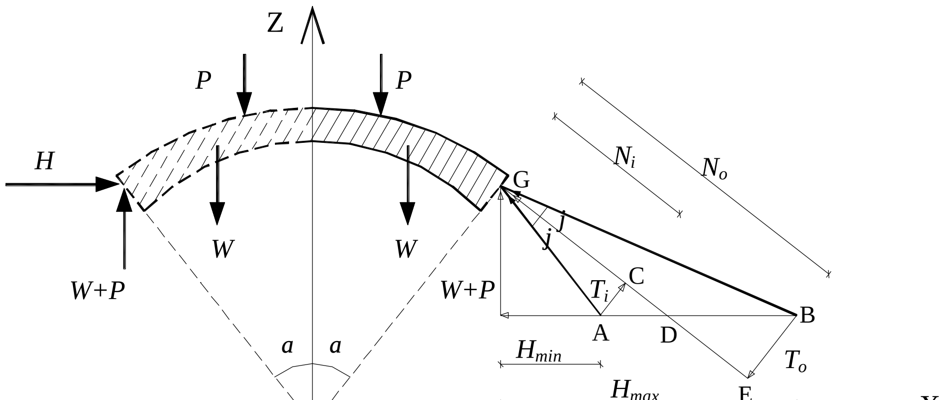

Fig. 2. The internal forces at a block interface of the arch.

In both cases of sliding, the limiting compressive and shear forces are statically defined and particularly, taking into account (1) follows that:

$$N_i = \frac{(W+P)}{\sin(a+j)}\cos j \quad ; \qquad T_i = \frac{(W+P)}{\sin(a+j)}\sin j \tag{2}$$

$$N_o = \frac{(W+P)}{\sin(a-j)}\cos j \quad ; \qquad T_o = \frac{(W+P)}{\sin(a-j)}\sin j \tag{3}$$

for inward sliding, corresponding to point A and represented in Figure 2 by vectors CG and AC, and for outward sliding, corresponding to point B and represented by vectors EG and BE. For a given geometry, these limiting values of T and N are independent of each other and only depend on the external applied load and on the given value of the friction angle. This means that, in case of symmetric loading, there is a unique limiting value of the shear force, that is T_i or T_o depending on the direction of sliding, and the local equilibrium problem is at a limit state statically determined.
This derives from the fact that, even for non-standard materials, if normal forces are given at a limit state, conditions (2) and (3), then they can be ignored in defining the yield surface, and, consequently, the Coulomb's cone reduces to a circle in the plane of the shear forces [18]. The size of the circle obviously depends on the magnitude of the normal force N, but the imposition of the normality rule now does not imply dilatancy, and hence the solution, being equilibrated at the yield surface and not violating the kinematical constraints, is indeed the correct solution and is unique. Therefore the material constraints become now standard and the analysis falls within the framework of the classical plasticity theory.

3.2 Limit-state analysis of 3D masonry arches

The uniqueness prove shown in the previous section is here used to calculate, with a lower-bound approach, the minimum thickness of a 3D circular arch of given radius and width, subjected to its own weight and to a couple of vertical loads P at some distance y_P from the origin of axes. The procedure is developed for spreadsheet and a multipurpose mathematical programming solver is used to solve the problem of optimum. The load condition is symmetrical with respect to the middle plane YZ of the arch. The symmetric half arch is modelled as a system of n rigid voussoirs interacting by compressive and frictional joints and the mass is distributed along the centre-surface of mean radius R. Having defined the XZ plane as a plane of symmetry for the geometry of the arch and its own weight, the projection on this plane of the mean surface of the arch and the load P are depicted in Figure 3a) together with a hypothetical line of thrust and the resultants of each voussoir's weight, W_i. Weights are considered as independent of the thickness of the arch.

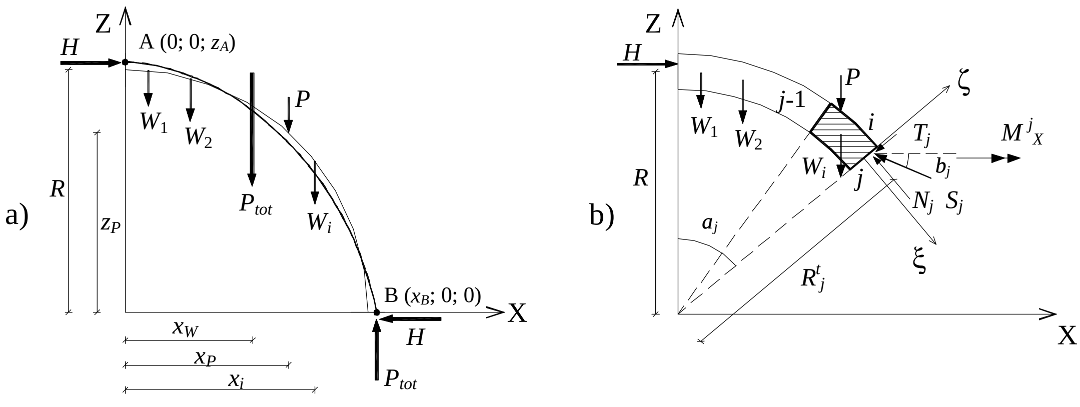

Fig. 3. a) Projection of the mean arch surface and hypothetical line of thrust on XZ.
b) Projection of the generic voussoir and interface stress resultants on XZ.

Coordinates x and y of the point of application of the load resultant $P_{tot} = P + \sum_{i=1}^{n} W_i$ are:

$$x_W = \frac{\sum_{i=1}^{n} (W_i \cdot x_i) + P \cdot x_P}{P_{tot}}; \qquad y_W = \frac{P \cdot y_P}{P_{tot}} \tag{4}$$

with the obvious meaning of the symbols.

Given the symmetry of the problem the unknown internal reactions are the thrust in the X direction *H*, and the resultant moment in the X direction M_X. The latter arises from the eccentricity in the *Y* direction of *P* and produces torsion at the interfaces and at the springs of the arch, resisted by friction. Equilibrium of rotation around the *Y* axis yields (Figure 3a)):

$$H = \frac{P_{tot}\left(x_B - x_w\right)}{z_A} \tag{5}$$

It is then possible to define a line of thrust of the arch, in the XZ plane, once two of its points are given, A(0; 0; z_A) and B(x_B; 0; 0) (e. g. with $z_A = x_B = R$) at the crown and the springing joints respectively. The unique solution of the minimisation problem is then defined by identifying the positions of A and B for which the correspondent thrust line yields the minimum thickness required, while satisfying the frictional constraints. For the minimisation problem z_A e x_B are assumed as the geometric unknowns. The solution is found with a discrete approach, by defining the geometry of the line of thrust through the calculation of its coordinates *x* and *z* at a number of points, for instance one for each voussoir. The generic coordinates of the line of thrust are:

$$x_i^t = x_i; \qquad z_i^t = z_{(i-1)}^t - \Delta^t z_i \qquad \text{for } i=1 \text{ to } n \tag{6}$$

In (6) x_i are the coordinates of the centre of gravity of each voussoir, while $\Delta^t z_i$ are obtained by considering the incremental equilibrium to rotation at each subsequent voussoir:

$$\Delta^t z_i = \frac{\sum_{k=1}^{i-1}\left(W_k + P_k\right)}{H}\,\Delta^t x_i \qquad \text{for } i=1 \text{ to } n \tag{7}$$

where P_i is the external load applied to the i^{th} voussoir from the crown and $\Delta^t x_i = x_i - x_{(i-1)}$.

Now considering the generic interface *j* (with *j*=0,...*n*), in Figure 3b), the resultant S_j forms with the X axis an angle b_j:

$$b_j = \arctan\frac{\sum_{k=1}^{j}\left(W_k + P_k\right)}{H} \tag{8}$$

The thrust line in (6) can then be described by its points on the interfaces by the following:

$$x_j^t = z_j^t \cdot \tan a_j; \qquad z_j^t = \frac{z_i^t + x_i^t \cdot \tan b_j}{1 + \tan a_j \cdot \tan b_j} \tag{9}$$

where a_j is the angle that the generic interface forms with the Z axis. Equations (9) can be used to calculate the distance between the origin of the axes and the point of the thrust line:

$$R_j^t = \sqrt{\left(x_j^t\right)^2 + \left(z_j^t\right)^2} \tag{10}$$

and hence the thickness *t*, with the hypothesis that is constant, is found by the relation:

$$t = \max\left|R - R_j^t\right| \tag{11}$$

that will be the objective function to minimise.

The material constraints are defined by limiting the maximum values of the internal shear force and torsional moment to be not greater than the frictional strength at each block interface. The resultant of internal forces and moments for the generic interface are:

$$S_j = \frac{\sum_{k=1}^{j}\left(W_k + P_k\right)}{\sin b_j} \qquad \text{and} \qquad M_X^j = M_X^{j-1} + P_i \cdot y_P^i \tag{12}$$

where $M_X^j = 0$ for the interfaces above the loaded voussoir, and constant for all the others. The components of these two vectors along the local axes ξ and ζ in Figure 3b) are:

$$N_j = S_j \cdot \cos(a_j - b_j); \qquad T_j = S_j \cdot \sin(a_j - b_j) \tag{13}$$

$$M_x^j = M_X^j \cos a_j; \qquad M_z^j = M_X^j \, sin a_j \tag{14}$$

Considering the generic interface *j* of dimensions *w* and *t* (arch's width and thickness), as shown in Figure 4, with local axes η and ζ (η parallel to global Y). The point C of application of N_j has eccentricity h_N^j, with respect to the centre of gravity of the surface:

$$h_N^j = \frac{M_z^j}{N_j} \tag{15}$$

If the shear resultant T_j is also applied in C, then the resultant torsion on the surface is:

$$M_t^j = M_x^j - T_j \, h_N^j \tag{16}$$

A reduced surface can then be considered over which it is assumed that the direct stresses s_N are uniformly distributed, with C centre of gravity and reduced dimensions a_j and b_j:

$$a_j = 2\left(\frac{w}{2} - \left|h_N^j\right|\right); \qquad b_j = 2\left(\frac{t}{2} - \left|R - R_j^t\right|\right) \tag{17}$$

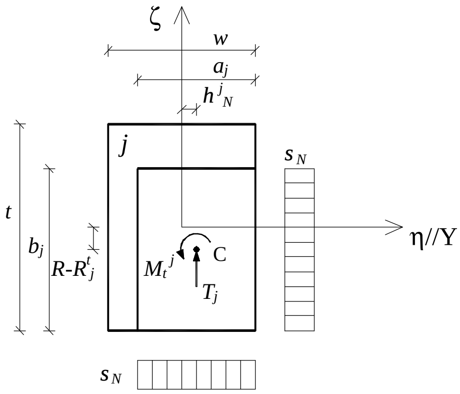

Fig. 4. Reduction of the active interface surface for interaction of torsion and shear.

This assumption is in favour of safety and does not alter the conditions set in the first section of the paper, which ensure the validity of the standard approach. Casapulla [19] has proposed an expression for the limit capacity of frictional surface in torsion in presence of shear parallel to one of the axis of symmetry, which is, again, in favour of safety:

$$M_{t(\lim)}^j = M_0^j \left(1 - \frac{\left|T_j\right|}{T_{\lim}^j}\right) \tag{18}$$

where T_j is given by (13), ($T_{\lim}^j = N_j \tan j$) and:

$$M_0^j = \frac{T_{\lim}^j}{12 a_j b_j}\left[a_j^3 \ln \frac{b_j + \sqrt{a_j^2 + b_j^2}}{a_j} + b_j^3 \ln \frac{a_j + \sqrt{a_j^2 + b_j^2}}{b_j} + 2 a_j b_j \sqrt{a_j^2 + b_j^2}\right] \tag{19}$$

is the limit frictional moment in absence of shear force. Therefore the problem of minimum, with variables x_B and $z_{A,}$ assumes the following standard format:

Minimise $t = \max\left|R - R_j^t\right|$ (20)

under the material constraint conditions:

$$\left|T_j\right| \le T_{\lim}^j \qquad \text{and} \qquad \left|M_t^j\right| \le M_{t(\lim)}^j \tag{21}$$

4. DISCUSSION OF RESULTS AND CONCLUSIONS

The procedure outlined above has been applied to the study of a simple 3D arch, made out of 27 voussoirs subjected to their own weight and to a couple of symmetrical point loads acting at a point at 60° from the vertical. The parameters that influence the minimum thickness are the coefficient of friction, the magnitude of the loads and their eccentricity with respect to the XZ plane of symmetry. The results are plotted for increasing value of the ratio of the concentrated load to the self weight of the half arch P/W_{tot}.

Figure 5 shows the effect of increasing the ratio between eccentricity and width for increasing values of P/W_{tot}. The friction coefficient has been set to 0.6, so as to prevent sliding mechanisms associated with the self-weight, and the width of the arch equals 10% of the radius. The curves show that for values of the vertical loads up to 20% of the half arch weight, the eccentricity to width ratio is uninfluential on the minimum thickness, except for values close to 1. For increasing values of the ratio P/W_{tot} the range over which the eccentricity is uninfluential decreases sharply, after which the relationship between the two ratios is linear. It is however interesting to notice that while for small values of the eccentricity the load placed at 60 degrees has a beneficial effect on the minimum thickness required (as it is well known for infill action), the effect becomes negative as the eccentricity increases.

The influence of the width of the arch on the minimum thickness, for a given value of the eccentricity produces further understanding into this behaviour. Keeping the eccentricity equal to 10% of the radius, the initial values of the curves in Figure 6 are equivalent to the values obtained in Figure 5 for *ecc./width = 1*, but, increasing the width, this ratio proportionally decreases, and so also the thickness required. The beneficial effect of the increasing width on the required thickness is more pronounced for smaller values of the live to dead load ratio. It is worth mentioning that these results are influenced by the particular model adopted for the interaction between torsional moment and shear for the resisting cross section. On the converse if the eccentricity is increased proportionally with the width of the arch the effect is detrimental. Figure 6 can be used both for design and assessment. In the first instance given a value of P/W_{tot} the minimum values of t and w can be identified, hence minimising the required structural dimensions. In case of assessment, provided a choice of safety factor for the applied load, couples of values of t and w above the chosen curve are safe, points below the chosen curve are unsafe conditions.

Finally, Figure 7 shows the increase in carrying capacity of the arch as the friction coefficient increases. Actually, when the eccentricity is taken as half of the width, and the width is 10% of the radius, for small values of P the minimum thickness is independent of the friction coefficient, while as the value of P increases the relationship becomes hyperbolic and, as expected, the minimum thickness increases for decreasing values of the friction coefficient.

On the basis of these results it can therefore be stated that the procedure proposed allows the plastic analysis of 3D masonry arches within the framework of the standard limit analysis in condition of eccentricity of load. This represents an enhancement with respect to the current state of the art of two-dimensional approaches. Furthermore the proof of uniqueness and the static approach ensure the robustness and safety of the solution obtained. However it is of interest to study the associated mechanisms, to identify which range of the parameters lead to sliding failure, in preference to hinging. This part of the study is currently under development.

The procedure presented can easily be extended to different loading condition and to the case of variable thickness. A further development, which is currently in progress, involves extension to the study of block masonry domes.

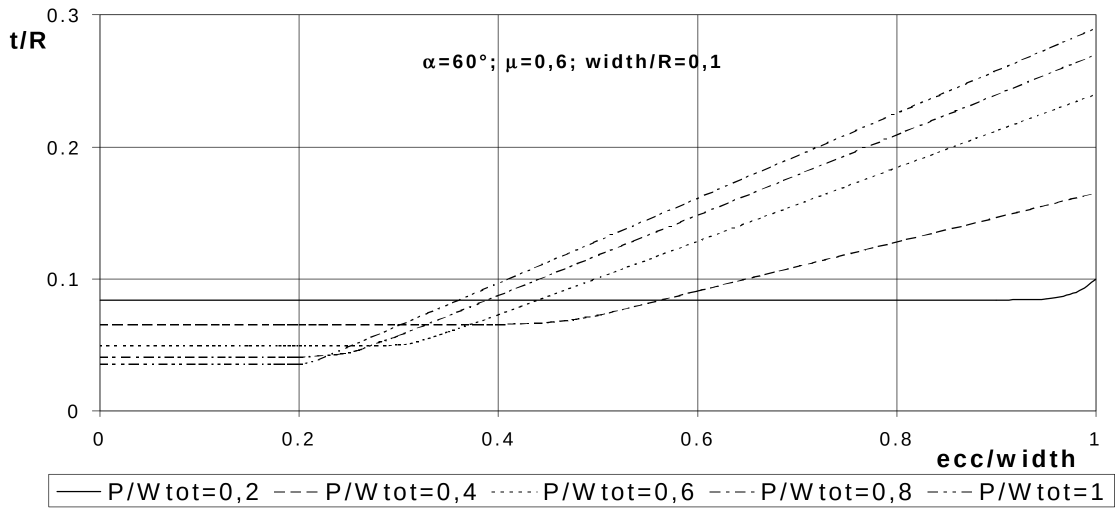

Fig. 5. The arch minimum thickness as a function of the load eccentricity.

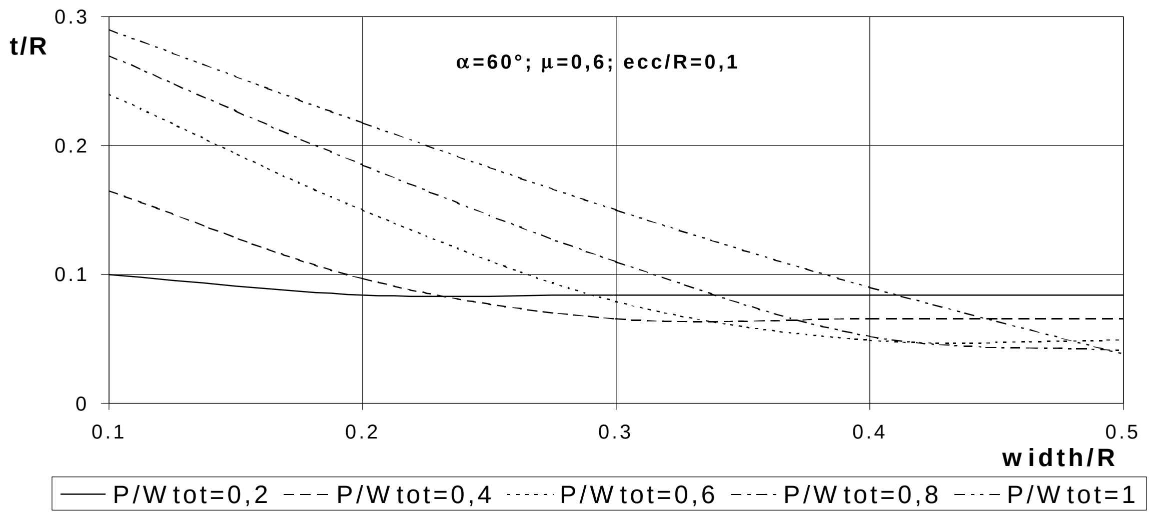

Fig. 6. The arch minimum thickness for constant eccentricity and variable width.

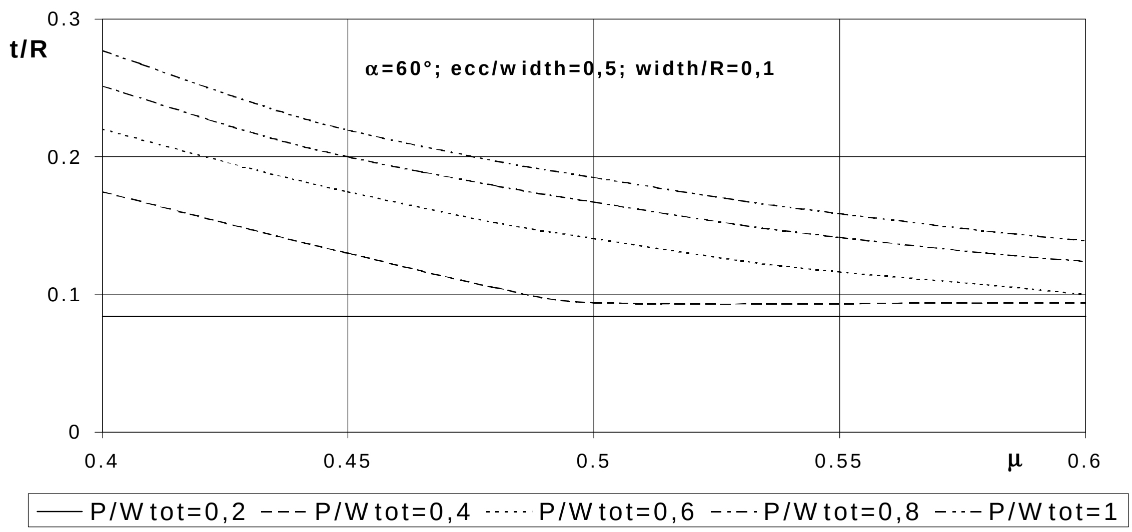

Fig. 7. The arch minimum thickness as a function of the friction coefficient.

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